J. Appl. Maths Mechs, Vol. 55, No. 2, pp. 273-276, 1991
Printed in Great Britain

0021-8928/91 \$15.00+0.00

OPTIMIZATION OF IRRIGATION IN A HYDRODYNAMIC FILTERING MODEL*

A.R. KASIMOV

An extremum of the cross-sectional area of a flooded or drained region in the context of the intake of a ground flow by a free-surface due to seepage is determined. The contour of perfect drainage of a maximum volume with recharge from a flooded ground surface is found. The boundary control of a channel for a viscous fluid flow is established.

In filtering theory, the characteristics of flows are determined using isoperimetric inequalities based on the optimization of the shape of the area of flow /1/. In what follows, problems of this type are solved by the method used in /2/ by three schemes: seepage above a head level, seepage to a perfectly drained level, and differential shear flow in a Taylor-Richardson cell.

We consider the following schemes for a two-dimensional flow with a vertical axis of symmetry: 1) seepage with irregular intensity and discharge $2q$ in a field BC of width $2L$, with a head level at depth H below the depression curve (Fig.1); 2) feeding with rate $2q$, under the effect of a pressure H of selfcontained drainage of BC to a confining bed lying at a depth T below the flooded ground surface (Fig.2); 3) viscous fluid flow in a channel of width $2L$ above a curvilinear base BC with a boundary AD moving with velocity U_0 which is subject to a resistance F_0 (Fig.3). In schemes *1* and *2*, the filtration is assumed to be linear and the ground is uniform and isotropic and in scheme *3*, we assume the hypotheses of /3, 4/. Then, the potential $\varphi(x, y)$ for schemes *1*, *2* and the current velocity $u(x,y)$ for scheme *3* satisfy the Laplace equation in the corresponding area of flow G_z.

We formulate the problem of optimising the boundary of G_z.

Problem 1. For given q, L, H, determine the shape of BC which gives an extremum of the area S of the zone of flooding beneath the field of the feed.

Problem 2. For given q, T, H, determine the drainage contour with an external volume S.

Problem 3. For given F_0, U_0, L determine the shape of the drainage BC such that the cell S is extremal.

We assume that the filtering coefficient is equal to unity and introduce the characteristic functions: 1) the Zhukov function $\omega = z + iw$, 2) the complex potential $w = \varphi + i\psi$, 3) the complex potential $p = u + iv$. In the planes ω, w, p (respectively), the area G_z will correspond to: 1) bands, 2) and 3) rectangles, which we denote by G_1, G_2, G_3. We map G_i into the half-plane $\operatorname{Im}\zeta > 0$ of the variable $\zeta = \xi + i\eta$ by the functions

$$\omega = -\frac{H}{\pi}\ln\frac{a+\zeta}{a-\zeta}, \quad a = \operatorname{cth}\frac{\pi(L-q)}{2H} \tag{1}$$

$$w = iqF(\arcsin\zeta, \lambda)/K, \quad q/H = K/K', \quad a = 1/\lambda$$

$$p = -iF_0F(\arcsin\zeta, \lambda)/(2K), \quad F_0/U_0 = 2K/K'$$

$$K = F(\pi/2, \lambda), \quad K' = F(\pi/2, \lambda'), \quad \lambda' = \sqrt{1-\lambda^2}$$

where F is an elliptic integral of the first type with modulus λ.

In scheme *1*, the function $w^* = -iw$, and in schemes *2* and *3*, the function $z = x + iy$, are analytic in the half-plane $\operatorname{Im}\zeta = 0$ and the boundary conditions for these have the form: 1) $\Psi = \operatorname{Im} w^* = 0$ for $|\xi| \geqslant a$, $\Phi = \operatorname{Re} w^* = q$ for $-a \leqslant \xi \leqslant -1$, $\Phi = -q$ for $1 \leqslant \xi \leqslant a$, $\Phi = f[\operatorname{Re}\omega(\xi)]$ for $-1 \leqslant \xi \leqslant 1$; 2) $y = 0$ for $|\xi| > a$, $y = -T$ for $1 \leqslant |\xi| < a$, $y = -T + y[\psi(\xi)]$ for $|\xi| \leqslant 1$; 3) $y = 0$ for $|\xi| \geqslant a$, $x = \pm L$ for $1 \leqslant |\xi| \leqslant a$, $x = x[v(\xi)]$ for $|\xi| \leqslant 1$. Then the solution of the mixed boundary-value problem for scheme *1* and the control function are given by:

$$w^* = -\frac{q}{\pi}\sqrt{\zeta^2 - a^2}\left[\int_{-a}^{-1} R(\tau,\zeta)\,d\tau - \int_{1}^{a} R(\tau,\zeta)\,d\tau - \int_{-1}^{1} \tau R(\tau,\zeta)\,d\tau + \int_{-1}^{1} \Omega(\tau) R(\tau,\zeta)\,d\tau\right] \tag{2}$$

**Prikl.Matem.Mekhan.*,55,2,338-341,1991

$$R(\tau,\ \zeta) = 1/[\sqrt{a^2-\tau^2}\,(\tau-\zeta)],\ f\,[\mathrm{Re}\,\omega\,(\xi)] = -q\xi + q\Omega\,(\xi),\quad \Omega\,(\pm 1) = 0$$

where, the $\sqrt{\zeta^2 - a^2}$ is taken to be positive for $\zeta > a$.

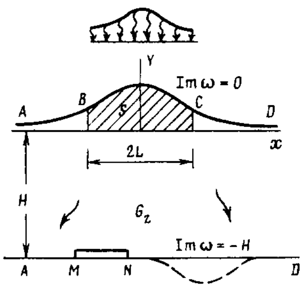

Fig.1

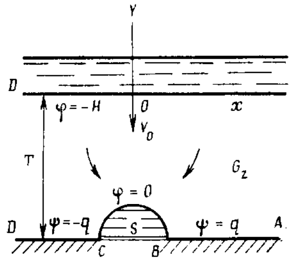

Fig.2

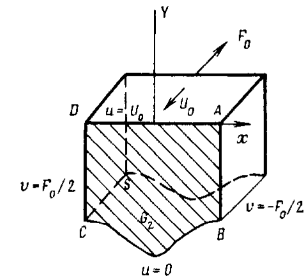

Fig.3

The solution of the previous Dirichlet problem for scheme 2 has the form

$$z = \frac{1}{\pi}\int_{-1}^{1}\frac{y\,[\psi(\tau)]}{\tau-\zeta}\,d\tau - \frac{T}{\pi}\ln\frac{a-\zeta}{a+\zeta} + x\,(\infty) \tag{3}$$

where $x(\infty) = 0$ in the case of symmetry about the y axis.

For scheme 3, the integral representation is identical with (2) if we replace w^* by z, q by L and Ω by X, and the control may be written as follows: $x\,[v\,(\xi)] = L\xi - LX\,(\xi),\ X\,(\pm 1) = 0$.

We search for solutions of Problems 1-3 in the class of functions $f\,[\mathrm{Re}\,\omega]\ y\,[\psi],\ x\,[v]$ satisfying the Hölder condition. We represent the control in (2), (3) in the form of series such that the area functionals are represented as quadratic forms in the coefficients of these series and possible restrictions are represented as linear forms. Thus, if in (2) we pass to the limit as $\zeta \to \xi \in [-1, 1]$ and make the substitution /5/ $\xi = a\cos\theta/\sqrt{a^2-\sin^2\theta},\ 0 \leqslant \theta \leqslant \pi$ writing $\Omega\,[\xi\,(\theta)] = \sum a_{2n}\sin 2n\theta$ (summing over n from 1 to ∞), then from the parameters of the control area BC of the depression curve, we obtain

$$\frac{S}{q^2} = -\int_{-1}^{1} y\,(\xi)\,x'\,(\xi)\,d\xi = \pi\sum n a_{2n}^2 + S_0 + a\,(1-a^2)\sum a_{2n}\,(J_{2n} + P_{2n}) \tag{4}$$

$$S_0 = \frac{1}{\pi}\int_{-1}^{1} y_0\,(\xi)\,x_0'\,(\xi)\,d\xi,\quad P_{2n} = \int_{0}^{\pi}\frac{\cos 2n\theta\sin\theta}{(a^2-\sin^2\theta)^{3/2}}\left[1 + \frac{2H}{\pi q}\,\frac{a^2-\sin^2\theta}{a\,(a^2-1)}\right]d\theta$$

$$x_0\,(\xi) = -\xi - \frac{H}{\pi q}\ln\frac{a+\xi}{a-\xi},\quad y_0\,(\xi) = \frac{1}{\pi}\left[2\arcsin\frac{1}{a}\sqrt{a^2-\xi^2} + \ln\frac{A_1A_2}{a^2\,(1-\xi^2)} - \xi\ln\left(\frac{A_1}{A_2}\,\frac{1+\xi}{1-\xi}\right)\right],\quad A_{1,2} = \sqrt{a^2-1}\,\sqrt{a^2-\xi^2} + a^2 \mp \xi$$

The non-degenerate extremum of the area functional in (4) is a minimum and, as in /2/, we may write

$$(S/q^2)_{\min} = S_0 - a^2\,(1-a^2)^2\sum (P_{2n} + J_{2n})^2/(4\pi n) \tag{5}$$

for this.

The solution of Problem 3 differs from (5) by the replacement of J_{2n} by $-J_{2n}$, and in this case

$$S_0 = \int_{-1}^{1} y_0\,(\xi)\,d\xi,\qquad P_{2n} = \int_{0}^{\pi}\cos 2n\theta\sin\theta/(a^2-\sin^2\theta)^{3/2}\,d\theta$$

In scheme 2, we represent the control in the form of a series

$$y\,[\psi\,(\xi)] = \sum b_n U_n\,(\xi),\quad U_n = \sin\,(n\arccos\xi)$$

Then

$$S = \int_{-1}^{1}\{y\,[\psi\,(\xi)] + T\}\,x'\,(\xi)\,d\xi = -\frac{\pi}{2}\sum n b_n^2 + \frac{2Ta}{\pi}\sum b_{2n-1}E_{2n-1}$$

$$E_{2n-1} = \int_0^\pi \sin(2n-1)x \sin x (a^2 - \cos^2 x)^{-1} dx$$

Whence, it follows that a non-degenerate extremum is a maximum of Problem 2 with

$$(S/T^2)_{\max} = 2a^2\pi^{-3} \sum E_{2n-1}^2/(2n-1) = \pi^{-1} \ln (a/\sqrt{a^2-1}) \tag{6}$$

Eq.(5) may be used to study the subsidence of mounds of surface water, since the volume or the area of the zone where the natural level of surface water is perturbed determines the rate of subsidence both in hydrodynamic and in hydraulic models /7/. The area of this zone appears in the expression for the filtering rate in the bottom layer *AD*, which is obtained from the formula for the conformal transformation of bands near the area /8/.

The original Problems 1-3 may be modified, both by considering other flow schemes and by introducing additional restrictions or modifications of the objective functional. For example, if in scheme *1*, the line *AD* is a contour representing supply rather than discharge, (i.e. if we consider the evaporation from *BC*) then we must replace $L-q$ by $L+q$ in (1) and the - sign of S/q^2 in (4). If in scheme (2), the area G_z is rotated by an angle π about the point $(0, -T)$, and the flow line is replaced by the equipotential, and vice versa, Eq.(6) then gives an estimate of the area of the underground part of the impermeable dike flood bed.

If in Problem 1, we search for an extremum of the area S_1 amongst all curves *ABCD*, then passing to the limit as $\zeta \to \xi, 1 \leqslant |\xi| \leqslant a$ in (2) and equating the parts *AB* and *CD*, we obtain

$$S_1/q^2 = S/q^2 + 4H(\pi q)^{-1} \sum a_{2n} W_{2n} + S_0^*$$

$$S_0^* = \frac{4aH}{\pi q} \int_1^a \frac{y_0(\xi)}{a^2-\xi^2} d\xi, \quad W_{2n} = \int_0^\infty \frac{\exp(-2n\theta_0)\,\mathrm{sh}\,\theta_0}{\sqrt{a^2+\mathrm{sh}^2\theta_0}} d\theta_0$$

and the extremum is expressed as a series

$$(S_1/q^2)_{\min} = S_0 + S_0^* - \sum [2HW_{2n}(\pi q)^{-1} + {}^1/_2 a(1-a^2)(J_{2n} + P_{2n})]^2/(\pi n) \tag{7}$$

In problems associated with the washing out of flooded layers by drainage or with the transport of contaminants in water-supply wells, the rate of filtering is very important. Thus, in Problem 2 we introduce the additional restriction on the rate V_0 at a point *O* above the drainage. Since $V_0 = \lim \psi'(\xi)/x'(\xi)$ as $\xi \to \infty$, $\psi'(\xi)$ may be extracted from (1) and for $x'(\xi)$ in *AD*, according to (3), we have the power series

$$x' = 2a\pi^{-1}T/(a^2-\xi^2) - \sum nb_n(\xi - \sqrt{\xi^2-1})^n/\sqrt{\xi^2-1}$$

Consequently, $b_1 = 2aq/(KV_0) - 4Ta/\pi$ and the extremum is expressed in the form

$$\left(\frac{S}{T^2}\right)_{\max} = \frac{1}{\pi} \ln \frac{a}{\sqrt{a^2-1}} - 2\pi a^2 \left(\frac{H}{TK'V_0} - \frac{2}{\pi} - \frac{E_1}{\pi^2}\right)^2$$

The estimates obtained may be used for more complicated flow schemes if variational theorems are used /1/. If, for example, in scheme *1*, we introduce a boundary (Fig.1, the dotted line) or part of this boundary is assumed to be impermeable, then the depression curve rises, the areas *S* and S_1 increase and the corresponding values of (5) and (7) will be minorants but improper.

REFERENCES

1. GOL'DSHTEIN P.B. and ENTOV V.M., Numerical Methods in the Mechanics of Continuous Media, Nauka, Moscow, 1989.
2. IL'INSKII N.B. and KASIMOV A.P., Filtrational optimization of the shape of a ground channel by the method of inverse boundary problems. Izv. Akad. Nauk SSSR, MZhG, 3, 1984.
3. TAYLOR G.I., A model for the boundary condition of a porous material, Part I. J. Fluid Mech. 49, Pt.1, 1971.
4. RICHARDSON S.A., A model for the boundary condition of a porous material. J. Fluid Mech. 49, Pt.2, 1971.
5. PYKHTEYEV G.N., Regular flow-past curvilinear arcs in the restricted and unrestricted flow of an ideal incompressible fluid. PMM. 19, 4, 1955.
6. DAGAN G., Two solutions of free-surface flow in porous media. J. Eng. Mech. Div. Proc. ASCE, 93, 4, 1967.

7. RUDAKOV V.K., Spreading of arbitrarily located mounds of ground water in the presence of drainage. Vodn. Resursy, 4, 1989.
8. LAVRENT'YEV M.A. and SHABAT B.V., Methods of the Theory of Functions of a Complex Variable. Nauka, Moscow, 1967.

Translated by T.Z.

J. Appl. Maths Mechs, Vol. 55, No. 2, pp. 276-279, 1991
Printed in Great Britain

0021-8928/91 $15.00+0.00

THE CONDITIONS FOR THE SOLUTIONS IN ELECTROMAGNETOELASTICITY TO BE EQUAL TO ZERO*

S.D. KLYACHKO

The quasistationary antiplane deformation of a cylinder and the twisting of a solid of rotation in conjunction with an electric or magnetic field for non-linear materials are considered. The conditions which guarantee that the stresses, displacements, induction and potential are zero for any boundary conditions are analysed.

For the quasistationary antiplane deformation of a cylinder of circular cross-section made of a linear uniform transversely isotropic piezoelectric material with an unloaded contour, the specification on part of the contour of a constant electric potential, and on part of the contour of the conditions for matching in a vacuum, and also for certain other problems of the same type, it is well-known /1/ that the mechanical stresses are zero everywhere, while the deformations and displacements are proportional to the strength and potential of the electric field. Below we analyse the presence of such properties in a certain class of problems and some allied problems.

1. Formulation of the problem. Consider the quasistationary antiplane deformation of a cylinder (the derivatives with respect to time are zero), in conjunction with a plane electric field for a non-conducting neutral piezoelectric material. On the basis of a well-known analogy, all the results derived later also hold for the twisting of a solid of rotation in conjunction with an axisymmetrical electric field. For brevity we will only consider the antiplane deformation of a cylinder. The transverse cross-section of the cylinder is singly-connected or multiply-connected and is arbitrary.

Suppose x_1, x_2 and x_3 are orthonormalized coordinates, and the x_3 axis is parallel to the generatrix. Further $i = 1, 2$; the notation is that generally used. Suppose the material is such that a state is possible for which only $u_3 = u$; $\gamma_3 = \gamma_i$; $\sigma_{i3} = \sigma_i$; φ; E_i; D_i are non-zero (and are independent of x_3), i.e., it is possible to consider the antiplane deformation of the cylinder in conjunction with a plane electric field (/1-4/ etc.).

We will write the equations of the problem as follows:

$$\sigma_{i,i} = 0;\ \gamma_i = u_{,i};\ D_{i,i} = 0;\ E_i = \varphi_{,i} \tag{1.1}$$

(φ differs in sign from that usually employed).

We will represent the contour Γ in the form $\Gamma = \Gamma_\sigma + \Gamma_u = \Gamma_D + \Gamma_\varphi$. On Γ we specify mechanical and, simultaneously, electrical boundary conditions as follows:

$$\Gamma_\sigma\colon\ \sigma_i n_i = s_\sigma;\quad \Gamma_u\colon\ u = s_u \tag{1.2}$$

$$\Gamma_D\colon\ D_i n_i = s_D;\quad \Gamma_\varphi\colon\ \varphi = s_\varphi \tag{1.3}$$

The case when arbitrary additive constants appear in u and φ is not specially stipulated, and these constants are fixed in a trivial way.

In the antiplane deformation of a cylinder we will consider two classes of non-linear anisotropic non-uniform materials (bodies) - which we will call A and B. Suppose class A is

*Prikl.Matem.Mekhan.,55,2,341-344,1991